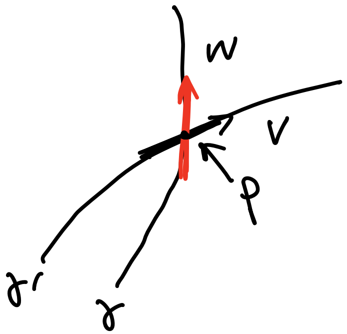
- Completeness of v.f. $X$

Given $X \in \Gamma(TM)$,

(1) every pt $p \in M$ lies in an integral curve; ← Due to fundemental Theorem in ODE

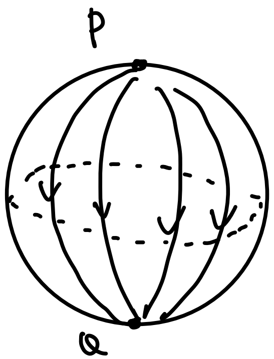
(2) any two integral curves do <u>not</u> intersect.




contradicts
$$\begin{array}{c} TM \\ \pi \downarrow \uparrow s \\ M \end{array}$$
is a map (so $p \to X(p)$ uniquely determined)

$\Rightarrow$ $M$ is "mostly" covered (or foliated) by integral curves

e.g.

$S^2 =$ 


$\xrightarrow{Z}$

$S^2 \setminus \{p, q\} = \bigsqcup$ (integral curves)

In general,

$$M \setminus \{\text{crit pts}\} = \bigsqcup \left( \begin{array}{c} \text{non-constant} \\ \text{integral curves} \end{array} \right)$$

Can we foliate $M \setminus \{\text{crit pts}\}$ by other high-dim'l subsets?

<u>Def</u> $X \in \Gamma(TM)$ is <u>complete</u> if every integral curve $\gamma$ is defined over $I = \mathbb{R}$.

(In other words, one can go along $\gamma$ for $t \to \pm\infty$).

<u>Prop</u> Any smooth v.f. on a cpt mfd is complete.

e.g. $X \in \Gamma(T\mathbb{R})$ where $X(p) = p^2 \partial_p$. Then any integral curve $\gamma : I \to \mathbb{R}$ satisfies

$$\dot{\gamma}(t) = \dot{p}(t) = p(t)^2$$

$$\Rightarrow \gamma(t) = \frac{-1}{t} + c \quad \text{so } I \neq \mathbb{R} \quad (\text{and } X \text{ is not complete}).$$

- Time-dependent vector field $X_t$ $(t \in I)$

e.g. $X_t(\underbrace{\rho, \theta}_{\text{polar coordinate of } \mathbb{R}^2}) := \left(\frac{1}{t}\cdot\rho, 0\right) \left(= \frac{1}{t}\rho\,\partial_\rho + 0\cdot\partial_\theta\right)$ $t \in (0,\infty)$

Any integral curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$, satisfying $\dot\gamma(t) = X_t(\gamma(t))$

$$(\dot\rho(t), \dot\theta(t)) = \left(\frac{\rho(t)}{t}, 0\right)$$

$$\Longrightarrow \begin{cases} \dot\rho(t) = \dfrac{\rho(t)}{t} & \text{solves } \rho(t) = \dfrac{t\rho(0)}{t_0} \\ \dot\theta(t) = 0 & (\text{so } \theta(t) = \theta(0) \text{ constant}) \end{cases}$$

Linear (cf. $\rho(t) = e^t \rho(0)$ in time-ind case).

③ <u>One parameter family of diffeos</u>

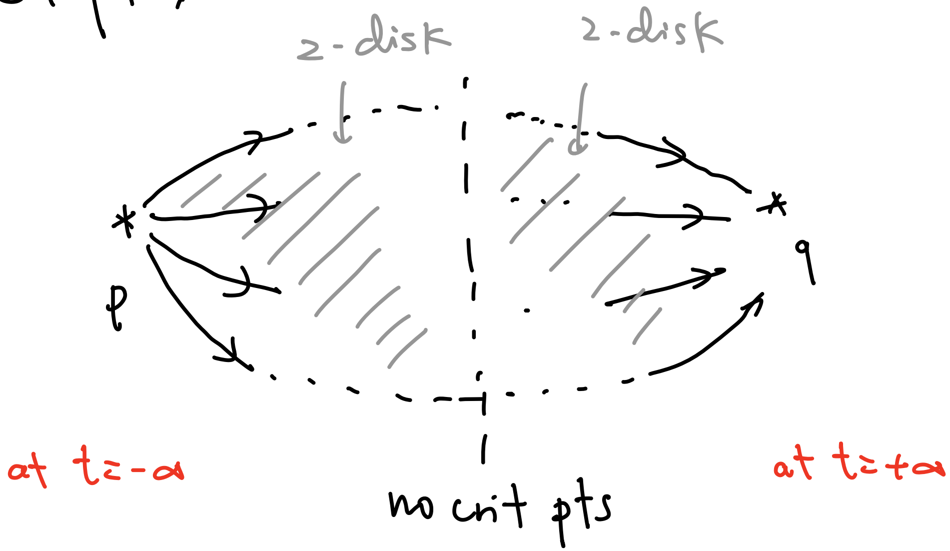
- Given $X \in \Gamma(M)$, assuming complete. define for $t \in \mathbb{R}$,

$$p \in M\left(=\{\text{critical pts}\}\cup\left\{\begin{matrix}\text{non-const.}\\ \text{int. curves}\end{matrix}\right\}\right) \xrightarrow{\ \varphi_X^t\ } \gamma(t_*+t)\in M$$

$$p=\gamma(t_*) \quad \text{some } \gamma,\ t_*$$

$$\left.\begin{matrix} -\ \ p\in\{\text{Crit pts}\}, \ \text{then} \ \ p\to p \ \ \forall t \\ -\ \ p\in\left\{\begin{matrix}\text{non-const.}\\ \text{int. curves}\end{matrix}\right\} \ \text{then} \ \ p\to p'(\neq p)\end{matrix}\right\} \ \varphi_X^t \text{ is a diffeo on } M \text{ for any } t.$$

e.g. Suppose a 2-dim $M$ admits a vector field $X$ has only 2 critical pts,




$$D^2 \cup_{\text{gluing}} D^2 \underset{\text{top}}{=} S^2$$

(homeo)

**Def** A one-parameter group of diffeo is a group homomorphism

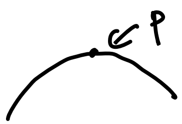
$$\Phi : \mathbb{R} \longrightarrow \mathrm{Diff}(M) \qquad t \longmapsto \varphi_t$$

**Prop.** A one-par group of diffeo $\Longleftrightarrow$ a vector field

**pf** "$\Longleftarrow$" as above

"$\Longrightarrow$" Given $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$, consider derivative wrt $t$:

$$\left.\frac{d\varphi_t}{dt}\right|_{t=0}(p) =: X(p) \quad \left( = \text{tangent vector of curve} \right)$$




$$= \{\varphi_t(p)\}_{t\in(-\varepsilon,\varepsilon)}$$

**Rmk.** Prop above endows vector fields a geometric meaning

**Rmk** (Palis) $\{\varphi \in \mathrm{Diff}(M) \mid \varphi = \varphi_1 \text{ for a one-par group of diffeo}\}$ is of first cat in $\mathrm{Diff}(M)$

vector fields generate few diffeo

__Rmk__ The vector field $X$ defined above satisfies

$$\frac{d\varphi_t}{dt}(p) = X(\varphi_t(p)) \longleftarrow \text{A famous formula}$$

Indeed, for any $t_*$, $\left.\frac{d\varphi_t}{dt}\right|_{t=t_*}(p) = \left.\frac{d(\varphi_{t-t_*}\circ\varphi_{t_*})}{dt}\right|_{t=t_*}(p)$

$$= \left.\frac{d\varphi_t}{dt}\right|_{t=0}(\varphi_{t_*}(p)) = X(\varphi_{t_*}(p)).$$

__Exe__ Let $\{\varphi_{s,t}\}_{(s,t)\in\mathbb{R}^2}$ be a 2-par group of diffeos, and

Here, one can simply denote X_{s,t} by X_s.

Here, one can simply denote Y_{s,t} by Y_t.

$$\frac{\partial\varphi_{s,t}}{\partial t} = X_{s,t}\circ\varphi_{s,t} \quad \text{and} \quad \frac{\partial\varphi_{s,t}}{\partial s} = Y_{s,t}\circ\varphi_{s,t}$$

Then prove $\frac{\partial X_{s,t}}{\partial s} - \frac{\partial Y_{s,t}}{\partial t} = [X_{s,t}, Y_{s,t}]$

e.g. Take $\Phi = \{\varphi_s\}_{s\in\mathbb{R}}$ and $\Psi = \{\psi_t\}_{t\in\mathbb{R}}$, and $X^{\Phi}$, $X^{\Psi}$.

Consider $\varphi_{s,t} := \psi_t \circ \varphi_s$

Then $\dfrac{\partial \varphi_{s,t}}{\partial t} = \dfrac{\partial(\psi_t \circ \varphi_s)}{\partial t} = X^{\Psi} \circ \psi_t \circ \varphi_s = X^{\Psi} \circ \varphi_{s,t}$

so $X_{s,t} = X^{\Psi}$.

Suppose $\psi_t \circ \varphi_s = \varphi_s \circ \psi_t$, so $\varphi_{s,t} = \varphi_s \circ \psi_t$. assumption

Then $\dfrac{\partial \varphi_{s,t}}{\partial s} = \dfrac{\partial(\varphi_s \circ \psi_t)}{\partial s} = X^{\Phi} \circ \varphi_s \circ \psi_t = X^{\Phi} \circ \varphi_{s,t}$

so $Y_{s,t} = X^{\Phi}$

Conclusion: two 1-par families of diffeomorphisms commute (in the sense of composition) if and only if their corresponding vector fields ``commutes'' in the sense that their bracket is equal to 0.

Then Exe above says that

$$[X^{\Psi}, X^{\Phi}] = [X_{s,t}, Y_{s,t}] = \frac{\partial X_{s,t}}{\partial s} - \frac{\partial Y_{s,t}}{\partial t} = 0$$

Rmk The converse also holds: $[X^{\Psi}, X^{\Phi}] = 0 \Rightarrow \psi_t \circ \varphi_s = \varphi_s \circ \psi_t$.